Journal of Applied Mathematics and Mechanics 73 (2009) 449–456

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The layered element method in the dynamic theory of elasticity☆

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ARTICLE INFO

Article history:
Received 2 July 2008

ABSTRACT

A semi-analytical approach is proposed for the numerical analysis of the dynamic behaviour of elastic layered systems and waveguides with internal and surface nonuniformities. The approach is based on representing the reflected field in the form of an expansion in fundamental solutions for the layered structure as a whole. Unlike classical boundary elements, which represent the fundamental solutions for a homogeneous elastic space, layered elements identically satisfy the boundary conditions on all outer and inner plane-parallel boundaries. Hence, to find the unknown expansion coefficients it is necessary to satisfy boundary conditions only on obstacles, that assumes the use of many fewer elements than in the boundary element method, using which it is necessary to arrange them along all the outer and inner boundaries of the region considered. Moreover, each layered element exactly describes the wave structure of the solution, which is particularly convenient when solving problems of the transmission and reflection of travelling waves in open waveguides with obstacles. A brief description of an algorithm for constructing layered elements is given and examples of the use of the layered element method for solving two-dimensional and three-dimensional problems of diffraction in an open waveguide, and also for calculating the dynamic characteristics of bounded structural elements, made of laminated materials, are presented.

1. Introduction

The need to solve dynamic problems for laminated elastic solids and waveguides with localized inhomogeneities arises both in engineering (for example, when designing structural components, made using modern composite materials), and geophysics, physical acoustics, flaw detection, the study of materials, medical tomography, etc. Methods of solving such problems are well known, based on the expansion of the required functions (wave fields) in terms of fundamental solutions, which identically satisfy the equations considered, so that by choosing the unknown expansion coefficients it only remains to satisfy the conditions on the boundary. The dimension of the problem can thereby be reduced by one, which is a definite advantage compared with approaches based on direct grid subdivision of the initial boundary-value problem, for example, the finite element method or the finite difference method.

This advantage becomes particularly noticeable when high-frequency (short-wave) oscillations are considered, requiring a considerable reduction in the size of the grid to approximate the wave motion even in regions with homogeneous properties of the elastic medium, whereas the fundamental solutions accurately describe the wave structure of the solution at inner points of the homogeneous region irrespective of the frequency. Moreover, the fundamental solutions satisfy in their structure the radiation conditions at infinity, whereas the use of the finite element and finite difference methods to calculate open (infinite) waveguides requires an artificial limitation of the region of subdivision by the introduction of absorbing boundary conditions or special infinite elements.

There are various approaches to the use of expansions in fundamental solutions. The first is the method of fundamental solutions when the required solution (the reflected wave field) is approximated in the region considered by a linear combination of fundamental solutions with centres (sources) arranged on the outside along its boundary. The unknown expansion coefficients are then determined from the condition for minimizing the discrepancy of the boundary conditions either by the simple collocation method, or using the Galerkin - Petrov projection technique.[1] In the second, the matrices of the fundamental solutions are used as the kernel of the boundary integral equations or as the basis of the boundary element method, which in this context can be regarded as the result of the discretization of the boundary integral equations.[2]

☆ *Prikl. Mat. Mekh.* Vol. 73, No. 4, pp. 622–634, 2009.
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doi:10.1016/j.jappmathmech.2009.08.005

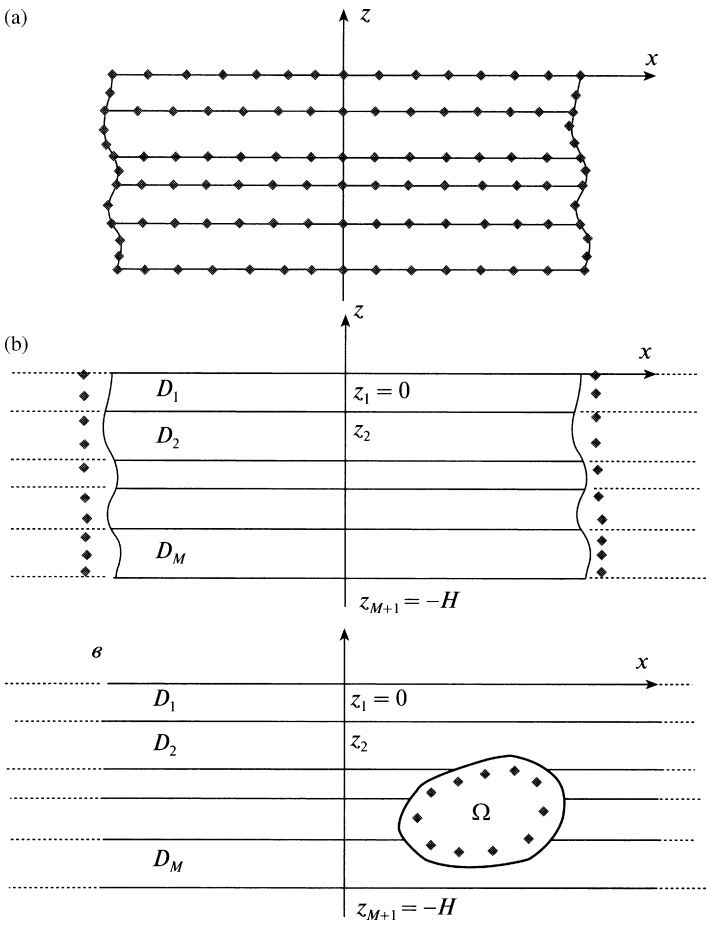

Fig. 1.

However, when considering laminated structures the use of the boundary integral equations or the boundary element method is not very convenient due to the need to place the elements not only along their outer boundaries but also along all the inner boundaries (Fig. 1a) (the centres of the elements are indicated by the points), since they do not satisfy the boundary conditions but only the equations for each of the subregions separately. This obstacle can be overcome if we take as the elements the fundamental solutions for the stratified structure considered as a whole, so that the conditions on the plane-parallel boundaries are satisfied identically (Fig. 1b). For example, comparatively simple semi-analytical representations of the fundamental solutions for orthotropic half-spaces and strips[3] were then used to derive the boundary integral equations on the surface of a depressed strip, which enabled the effective algorithm for solving the inverse problem of constructing its shape to be developed.[4] The kernel of the matrix boundary integral equations, used in previous papers[5,6] to solve problems of diffraction at arbitrarily oriented cracks, also identically satisfy the homogeneous boundary conditions on the outer boundary of an elastic half-space or layer.

Nevertheless, in the general case of a laminated medium, the absence of an explicit analytical representation of the fundamental solution (a laminated element) makes it difficult to use this approach, which appears more natural than the classical finite element method and the boundary element method, both when calculating bounded structural elements, made of laminated materials, and when solving problems of diffraction by localized obstacles in open elastic waveguides. Below, in order to construct laminated elements, we will use a modification of a numerically-stable method of constructing Green's matrix of a laminated elastic half-space.[7] A detailed description of the layered element method was given previously,[8] and hence in this paper we will only describe its general scheme; the features of specific forms will be illustrated by numerical examples for a bounded laminated sample, and also for two- and three-dimensional problems of diffraction by internal cavities and inclusions in an elastic waveguide. When checking the reliability of the results obtained using this method, in addition to numerical monitoring of the boundary conditions and the energy balance, we will compare the results obtained with existing results obtained by others.

2. The general scheme of the method

Starting from the fact that arbitrary non-stationary variables of a linearly elastic medium $\mathbf{u}(\mathbf{x}, t)$ can be represented in the form of the frequency superposition of steady harmonic oscillations $\mathbf{u}(\mathbf{x}, \omega)e^{-i\omega t}$, we will confine ourselves below to considering problems of the relatively complex amplitude of an harmonic wave field $\mathbf{u}(\mathbf{x}) \equiv \mathbf{u}(\mathbf{x}, \omega)$, omitting, as is usually done, the harmonic factor $e^{-i\omega t}$. To be specific, we will consider a free M-layered waveguide, the oscillations of the isotropic elastic layers of which satisfy Lamé's equations

$$(\lambda_n + \mu_n)\nabla \mathrm{div}\mathbf{u} + \mu_n \Delta \mathbf{u} + \rho_n \omega^2 \mathbf{u} = 0, \quad \mathbf{x} \in D_n, \quad n = 1, 2, \ldots, M \tag{2.1}$$

and the conditions that the layers are firmly stuck together

$$[\mathbf{u}]_n = 0, \quad [\boldsymbol{\tau}]_n = 0, \quad n = 2, 3, \ldots, M \tag{2.2}$$

Here λ_n, μ_n and ρ_n are Lamé's constants and the density of each of the component layers of the waveguide, which, in a Cartesian system of coordinates $\mathbf{x} = \{x, y, z\}$, occupy the regions

$$D_n : \infty < x, y < \infty, \; z_{n+1} \le z \le z_n, \; z_1 = 0, \quad z_{M+1} = -H$$

and H is the overall thickness of the waveguide. We denote the jumps in displacements $\mathbf{u} = \{u_x, u_y, u_z\}$ at the interfaces of the layers $z = z_n$ by $[\mathbf{u}]_n = \lim_{z \to z_n^-} \mathbf{u} - \lim_{z \to z_n^+} \mathbf{u}$ and similarly we denote the jumps in the stress field $\boldsymbol{\tau} = \{\tau_{xz}, \tau_{yz}, \sigma_z\}$ by $[\boldsymbol{\tau}]_n$.

The outer boundaries are assumed to be stress-free

$$\boldsymbol{\tau}|_{z=0} = 0, \quad \boldsymbol{\tau}|_{z=-H} = 0 \tag{2.3}$$

with the exception, possibly, of a region where an external load $\mathbf{q}$ is applied, which generates an initial field $\mathbf{u}_0$. When solving diffraction problems the field $\mathbf{u}_0$ is usually taken in the form of travelling waves, arriving from an infinitely distant source and incident on the obstacle considered (a crack, an inclusion, etc.).

Undoubtedly, other boundary conditions can also be considered, for example, rigid clamping of one of the outer boundaries: $\mathbf{u}|_{z=-H} = 0$, or non-rigid (spring) contact between the layers: $[\mathbf{u}]_n = b\tau|_{z=z_n}$. This only requires a small obvious modification of the algorithm for constructing the layered elements; the general scheme of the layered element method remains unchanged.

In the general case, the total field $\mathbf{u}$, which is formed in a bounded laminated sample or in an infinite (open) waveguide with localized inhomogeneities (defects), is made up of the specified incident field $\mathbf{u}_0$ and the unknown field of reflected waves $\mathbf{u}_{sc}$. Within the framework of the layered element method, $\mathbf{u}_{sc}$ is approximated by the sum of layered element fields, which is the product of the matrices of the fundamental solutions of the laminated structure $l_j(\mathbf{x})$ considered and column vectors of the unknown expansion coefficients $\mathbf{c}_j$:

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{u}_N(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \sum_{j=1}^{N} l_j(\mathbf{x})\mathbf{c}_j \tag{2.4}$$

The columns of the matrices $l_j(\mathbf{x}) = l(\mathbf{x}, \xi_j)$ are displacement vectors, produced in an infinite waveguide without obstacles by three (or, in the plane case, by two) linearly independent point sources, situated at the points ξ_j (the centres of the layered elements). By construction they thereby identically satisfy Eq. (2.1) and boundary conditions (2.2) and (2.3) at all points $\mathbf{x}$, with the exception of ξ_j. Hence, the boundary conditions on the obstacles are only approximated by choosing the coefficients $\mathbf{c}_j$. The centres ξ_j must then be arranged so that they do not violate the identical satisfaction of Eq. (2.1), i.e., they must be on the outside (or, in the limit, on the boundary) of the region in which the displacement field is approximated by expansion (2.4). For example, when considering a bounded sample the sources are situated on the outside (along its vertical ends in Fig. 1b), while for the field reflected by a cavity or an inclusion they must be situated inside the region Ω, occupied by the obstacle (Fig. 1c).

To satisfy the boundary conditions imposed on the stress field τ, we will use the following expansion, which arises from approximation (2.4),

$$\boldsymbol{\tau}_N(\mathbf{x}) = \boldsymbol{\tau}_0(\mathbf{x}) + \sum_{j=1}^{N} r_j(\mathbf{x})\mathbf{c}_j \tag{2.5}$$

in which the columns of the matrices r_j are the stress vectors, obtained by applying the stress operator

$$T(\mathbf{n}) : \boldsymbol{\tau} = T(\mathbf{n})\mathbf{u} \equiv \lambda \mathrm{div}\mathbf{u} \cdot \mathbf{n} + 2\mu\partial\mathbf{u}/\partial\mathbf{n} + \mu(\mathbf{n} \times \mathrm{rot}\mathbf{u})$$

to the column vectors l_j, where $\mathbf{n}$ is the outward normal to the surface on which the stresses τ are calculated.

Suppose $S = \partial\Omega$ is the boundary of the obstacle. For simplicity, the description given below is for a two-dimensional (plane) formulation ($\mathbf{u} = \{u_x, u_z\}$, $\tau = \{\tau_{xz}, \sigma_z\}$) and on the assumption that the displacements $\mathbf{w}$ are specified on part of the boundary S_u, and the stresses $\mathbf{p}$ are specified on S_τ:

$$\mathbf{u}|_{S_u} = \mathbf{w}, \quad \boldsymbol{\tau}|_{S_\tau} = \mathbf{p} \tag{2.6}$$

If the obstacle is a cavity or a crack with a stress-free inner surface, then $S_\tau = S$, $\mathbf{p} = 0$. The opposite situation $S_u = S$, $\mathbf{w} = 0$ occurs for fixed rigid inclusions or when the waveguide is rigidly clamped at the ends.

In many other cases the initial problem can also be reduced to a set of auxiliary problems with boundary conditions of the form (2.6). For example, for an inertial (fixed) rigid inclusion it is necessary to consider three auxiliary problems for determining the reaction of the medium to elementary shifts and rotations of the solid. In the case of an elastic inclusion, the deformation of which is approximated by finite elements, for the internal finite-element solution to be identical with the external expansion over the layered elements it is necessary to solve auxiliary problems for specified elementary displacements of each of the nodes lying on the boundary for zero displacements of the remaining ones etc.

The formulation of expansions (2.4), (2.5) under conditions (2.6) with subsequent discretization using the collocation method or by projection of the discrepancy onto the system of projectors φ_i $i = 1, 2, \ldots, N$) leads to a system of linear algebraic equations

$$B\mathbf{c} = \mathbf{f} \tag{2.7}$$

in terms of the generalized vector of the unknown expansion coefficients $c = \{c_1, c_2, \ldots, c_N\}$, where $f = \{f_1, f_2, \ldots, f_N\}$ is the generalized vector on the right-hand side of the system. The matrix $B = [B_{ij}]_{i,j=1}^{N}$ consists of blocks b_{ij} with dimensions of 2×2 (or 3×3 in the three-dimensional case), which are expressed in terms of the layered element matrix l_j or r_j.

For example, the requirement that boundary conditions (2.6) must be satisfied at the collocation points $\mathbf{x}_i \in S$ reduces to the form

$$b_{i,j} = \begin{cases} l_j(\mathbf{x}_i), & \mathbf{x}_i \in S_u \\ r_j(\mathbf{x}_i), & \mathbf{x}_i \in S_\tau \end{cases}, \quad \mathbf{f}_i = \begin{cases} \mathbf{w}(\mathbf{x}_i) - \mathbf{u}_0(\mathbf{x}_i), & \mathbf{x}_i \in S_u \\ \mathbf{p}(\mathbf{x}_i) - \boldsymbol{\tau}_0(\mathbf{x}_i), & \mathbf{x}_i \in S_\tau \end{cases} \tag{2.8}$$

Using the projection scheme, we have

$$b_{i,j} = \int_{S_u} l_j(\mathbf{x})\varphi_i(t)dt + \int_{S_\tau} r_j(\mathbf{x})\varphi_i(t)dt$$
$$\mathbf{f}_i = \int_{S_u} (\mathbf{w}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}))\varphi_i(t)dt + \int_{S_\tau} (\mathbf{p}(\mathbf{x}) - \boldsymbol{\tau}_0(\mathbf{x}))\varphi_i(t)dt \tag{2.9}$$

The integration here is carried out over the local coordinate t, defining the position of the current point $\mathbf{x}(t)$ on the boundary of the obstacle S. Since functions, localized in a limited section are usually chosen as the projectors (for example, traditional caps $\varphi_i(t) = \varphi(t - t_i)/h$ with a function of the form $\varphi(x) = 1 - |x|$, $|x| < 1$ then for small h, as in relations (2.8), only the integrals over S_u (when $\mathbf{x}(t) \in S_u$) or over S_τ (for $\mathbf{x}(t) \in S_\tau$) make a contribution to b_{ij} and $\mathbf{f}_i$. In the three-dimensional case the boundary S is the surface and correspondingly the integrals in formulae (2.9) are double integrals.

To use layered elements within the framework of the boundary integral equations method or the boundary element method, instead of expansions (2.4) and (2.5) the solution is sought in the integral form

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \int_S l(\mathbf{x}, \xi)\mathbf{c}(s)ds, \boldsymbol{\tau}(\mathbf{x}) = \boldsymbol{\tau}_0(\mathbf{x}) + \int_S r(\mathbf{x}, \xi)\mathbf{c}(s)ds; \quad \xi(s) \in S \tag{2.10}$$

in which $\mathbf{c}(s)$ is an unknown vector function. When integrating, the position of the centre of the fundamental solutions ξ is changed, i.e., actually the kernels l and r themselves, and hence these representations are traditionally used only if the kernel can be written in explicit form. However, when there are effective algorithms for constructing l and r it is quite possible to use them in the case of layered structures, especially as, in the case of discretization, it is necessary to calculate the values of the kernels only for a finite number of centres ξ_j and collocation points $\mathbf{x}_i$ or nodes of numerical integration in small cells $|t - t_j| < h$ and $|s - s_i| < h$.

For example, to derive expansions in layered elements from relations (2.10) we can use a piecewise-linear approximation of the unknown

$$\mathbf{c}(s) \approx \mathbf{c}_N(s) = \sum_j^N \mathbf{c}_j \varphi_j(s)$$

where ϕ_j are spline-caps, localized in the neighbourhood of the nodes s_j (see above), while the unknown coefficients of the expansion $\mathbf{c}_j$ are identical with its values at the nodes: $\mathbf{c}_j = \mathbf{c}(s_j)$. Substitution of $\mathbf{c}_N(s)$ into relations (2.10) again leads to expansions of the form (2.4) and (2.5), in which

$$l_j(\mathbf{x}) = \int_{|s-s_j| \le h} l(\mathbf{x}, \xi)\varphi_j(s)ds, \quad r_j(\mathbf{x}) = \int_{|s-s_j| \le h} r(\mathbf{x}, \xi)\varphi_j(s)ds \tag{2.11}$$

as before identically satisfy the boundary conditions on all plane-parallel boundaries of the packet of layers and the initial equations for all $\mathbf{x} \notin S_j$, where S_j is the part of the surface S corresponding to the change in the integration parameter in the interval $s - s_j < h$.

Integrals (2.11) can also be approximated by quadrature formulae, for example, for Gauss nodes or can even be replaced by average values at one node (for a sufficiently small step h). Here it is necessary to bear in mind that as $\mathbf{x} \to S_j$ the kernels become singular (more correctly, a singularity of the fundamental solutions l and r appears when the quantity ξ, which varies during integration, occurs with $\mathbf{x} \in S_j$). Since the principal (singular) part of the kernel, by construction, is identical with the classical fundamental solution for a point source in an unbounded homogenous space, here practically all the examples of the separation and integration of singular components, which were developed within the framework of the traditional boundary integral equation method and the boundary element method, remain in force.[2,9] We will also use an approach based on the special choice of the position of the nodes s_j and collocation points t_i, which ensure correct consideration of the singular components when $i = j$.[10]

On the whole, the different methods of approximation and discretization, that are possible within the framework of the layered element method, do not differ outwardly from the similar traditional schemes, based on classical fundamental solutions. Hence, the necessary condition for using it when solving specific problems is the availability of effective algorithms for constructing layered elements.

3. Integral representation of a layered element

When calculating the wave field $\mathbf{u}$, excited in a multilayered medium by a specified surface load $\mathbf{q}$, an integral approach, which uses a representation of $\mathbf{u}$ in the form of the convolution of Green's matrix of the medium k with $\mathbf{q}$ or in the equivalent form of an inverse Fourier transform of the product of their Fourier symbols K and $\mathbf{Q}$

$$\mathbf{u}(\mathbf{x}) = \int_{-\infty}^{\infty} k(x - \xi, z)\mathbf{q}(\xi)d\xi = \frac{1}{2\pi}\int_\Gamma K(\alpha, z)\mathbf{Q}(\alpha)e^{-i\alpha x}d\alpha \tag{3.1}$$

has been well recommended. The columns of the matrix k are solutions for the concentrated surface load applied along the coordinates of the unit vectors $\mathbf{i}_1 = \{1, 0\}$ and $\mathbf{i}_2 = \{0, 1\}$:

$$\boldsymbol{\tau}\,|_{z=0} = \delta(x)\mathbf{i}_k, \quad k = 1, 2 \tag{3.2}$$

(in the three-dimensional case along three unit vectors). The values of $\mathbf{u}(\mathbf{x})$ are determined by numerical integration or by replacing the contour integral by the sum of the residues at the poles of the elements of the matrix $K(\alpha, z)$. More details on the method of deriving this representation, choosing the contour of integration Γ, methods of constructing $K(a, z)$ and calculating $\mathbf{u}$ can be found in Refs 5–7, and the papers listed there.

Since the numerical realization of the integral approach is well developed and there is a stable algorithm for constructing the matrix $K(\alpha, z)$, similar integral representations and matrix algorithms for calculating their Fourier symbols can also be used to construct layered elements. The elements of the matrix $K(\alpha, z)$ are determined from a system of ordinary z-differential equations with piecewise-constant coefficients, and hence within the limits of each layer D_n its general solution and, correspondingly, the functions occurring in K, can be written in an explicit form with four unknown constants $t_n = \{t_n^{(1)}, t_n^{(2)}, t_n^{(3)}, t_n^{(4)}\}$, which are then found from the Fourier-transformed boundary conditions (2.2) and (2.3) with the first of conditions (2.3) replaced by (3.2). As a result of a relatively generalized vector of the unknown constants $\mathbf{t} = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_M\}$, the following algebraic systems of dimension $4M$ arise

$$A\mathbf{t} = \mathbf{f}_k, \quad k = 1, 2 \tag{3.3}$$

with the partitioned-diagonal matrix

$$A = \begin{Vmatrix} S_1(z_1) & 0 & 0 & \ldots & 0 & 0 \\ C_1(z_2) & -C_2(z_2) & 0 & \ldots & 0 & 0 \\ 0 & C_2(z_3) & -C_3(z_3) & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & C_{M-1}(z_M) & -C_M(z_M) \\ 0 & 0 & 0 & \ldots & 0 & S_M(z_{M+1}) \end{Vmatrix}$$

and the vectors of the right-hand side $\mathbf{f}_k = \{\mathbf{i}_k, 0, \ldots, 0\}$. The first and last two rows of the matrix A (blocks S_1 and S_M) follow from conditions (2.3) on the outer boundaries, while the groups of four rows with blocks $C_{n-1}(z_n)$ and $-C_n(z_n)$ ($n = 2, \ldots, M$) follow from conditions (2.2). The specific form of the general solution and of the 2×4 partitioned matrix S_m and the 4×4 partitioned matrix C_m are known (see, for example, Refs 7 and 8).

The matrix $l(\mathbf{x}, \boldsymbol{\xi})$, which specifies the layered element, differs from $k(\mathbf{x})$ in the fact that its columns are solutions for concentrated loads $\delta(\mathbf{x})\mathbf{i}_m$ applied at the internal point $\boldsymbol{\xi}$, rather than to the surface of the laminated packet. When constructing the Fourier symbol

$$L(\alpha, z, \xi) : l(\mathbf{x}, \xi) = \frac{1}{2\pi}\int_{\Gamma} L(\alpha, z, \xi)e^{-i\alpha x}d\alpha$$

this difference manifests itself only on the right-hand side of system $\mathbf{f}_m$, whereas the matrix A remains as before. Notably, the wave field of the source, situated in the n-th layer ($\boldsymbol{\xi} \in D_n$) is sought in the form of the sum of the fundamental solution for unbounded space $g(\mathbf{x}, \xi)\mathbf{i}_k$ (only in the limits of D_n) and the field $\mathbf{v}(\mathbf{x})$, which arises due to refractions from the layers. The matrix of the fundamental solutions g and its Fourier symbol $G(\alpha, z, \boldsymbol{\xi})$ are written in explicit form (see, for example, Ref. 7), while, relative to $\mathbf{v}$, after substituting $\mathbf{v} + g\mathbf{i}_k$ into boundary conditions (2.2) and (2.3) a boundary-value problem arises with non-uniform boundary conditions on the boundaries of the layer D_n: $z = z_n$ and $z = z_{n+1}$. If this is an internal boundary, the jumps of the field $\mathbf{v}$ and of the corresponding stresses $\sigma = T\mathbf{v}$ on this boundary suffer a discontinuity:

$$[\mathbf{v}]_n = -g\mathbf{i}_k\,|_{z=z_n}, \quad [\boldsymbol{\sigma}]_n = -Tg\mathbf{i}_k\,|_{z=z_n}$$
$$[\mathbf{v}]_{n+1} = g\mathbf{i}_k\,|_{z=z_{n+1}}, \quad [\boldsymbol{\sigma}]_{n+1} = -Tg\mathbf{i}_k\,|_{z=z_{n+1}} \tag{3.4}$$

while on the outer boundaries $z = z_1$ or $z = z_{N+1}$, the field $g\mathbf{i}_k$ induces stresses

$$\boldsymbol{\sigma}\,|_{z=0} = -Tg\mathbf{i}_k\,|_{z=0} \quad \text{или} \quad \boldsymbol{\sigma}\,|_{z=-H} = -Tg\mathbf{i}_k\,|_{z=-H} \tag{3.5}$$

The Fourier symbols on the right-hand sides of conditions (3.4) and (3.5) also give non-zero components in the corresponding parts of the vectors of the right-hand side $\mathbf{f}_k$.

In practice, a scheme for constructing layered elements that does not require using a matrix of the fundamental solutions g turned out to be more convenient. The idea of this is that a point discontinuity of the displacements or stresses at the interface of the layers is taken as the source:

$$[\mathbf{u}]_n = \delta(x)\mathbf{i}_k \quad \text{или} \quad [\boldsymbol{\tau}]_n = \delta(x)\mathbf{i}_k \tag{3.6}$$

Since, as a rule, the source $\boldsymbol{\xi} = \{\xi_1, \xi_2\}$ does not lie on the boundary of the layers, an additional boundary $z = \xi_2$ is constructed artificially, separating the homogeneous layer into two with the same properties. Formally, the packet becomes $(M+1)$-layered, but some increase in the numerical costs associated with this is compensated by the simplicity of the algorithm. As when constructing $K(\alpha, z)$, the vectors of the right-hand side $\mathbf{f}_k$ consist only of zeros and a single unity, the position of which is determined by the number n of the boundary $z_n = \xi_2$ introduced and the type of source. For a discontinuity in the stresses the unit vectors $\mathbf{i}_k$ appear in $\mathbf{f}_k$ at the level of the first two rows of the blocks $C_m(z_n)$, while for a discontinuity of the displacements they appear at the level of the last two. The algorithm then becomes uniform irrespective of whether the centre $\boldsymbol{\xi}$ is situated at internal points or lies on an interface. It is obvious that if $\boldsymbol{\xi}$ lies on the outer

surface $z = z_1$, then $\mathbf{f}_k = \{i_k, 0, \ldots, 0\}$ and the matrix of layered elements $l(\mathbf{x}, \boldsymbol{\xi})$ is identical with $k(\mathbf{x})$, and if it is on the lower surface, $z = -H$, then $\mathbf{i}_k$ is at the position of the last two components of the vector $\mathbf{f}_k$, which gives a matrix similar to $k(\mathbf{x})$, but for a surface load applied to the lower side of the packet. Such uniformity turned out to be convenient for using different methods of applying the layered-element method.

4. Numerical examples

A verification of the method. The general scheme of the layered element method was realized in several versions, differing in the methods of setting up the system (the collocation method, the Galerkin - Petrov scheme, and discretization of the boundary integral equations) and in the method of taking into account the singularity of the kernel for solving problems in the plane and three-dimensional formulation. To estimate their comparative effectiveness and to monitor the reliability of the results obtained, we mainly estimated the relative integral error ε in satisfying the approximated boundary conditions (2.6)

$$\varepsilon = \int_{S_u} \frac{|\mathbf{u}_N - \mathbf{w}|}{|\mathbf{u}_0 - \mathbf{w}|} dt + \int_{S_\tau} \frac{|\boldsymbol{\tau}_N - \mathbf{p}|}{|\boldsymbol{\tau}_0 - \mathbf{p}|} dt \tag{4.1}$$

(the discrepancy is attributed to the amplitude of the scattered field at the boundary).

For example, below we show how ε depends on the number of elements N when considering the diffraction of the field $\mathbf{u}_0$ by a fixed elliptic inclusion with boundary

$$S: (x - x_c)^2/a^2 + (z - z_c)^2/b^2 = 1$$

($x_c = 0$, $z_c = -0.5$, $a = 0.5$, $b = 0.1$, $\omega = 2$, $\mathbf{w} \equiv 0$ in the first condition (2.6)), which is in a homogeneous layer of unit thickness ($H = 1$), for discretization by the collocation method (ε_c) and using Galerkin's scheme (ε_G) when the sources are a distance of $\delta = |\mathbf{x}_{i+1} - \mathbf{x}_i|/N$ from the boundary

N	8	16	24	32	40
ε_c	0.2370	0.0344	0.0175	0.0190	0.0215
ε_G	0.0310	0.0130	0.0087	0.0057	0.0044

It can be seen that the projected scheme gives a higher accuracy than the collocation conditions at the nodes $\mathbf{x}_i$, situated on S_u half way between the projections of the centres of the sources ξ_j.

Here and henceforth the input parameters and the results are presented in dimensionless form in units expressed in terms of three characteristic quantities: the length l_0, the velocity v_0 and the density ρ_0. Here the dimensionless angular frequency $\omega = 2\pi f l_0/v_0$, where f is the dimensional frequency in Hertz. In the examples considered for a layer we chose the layer thickness H, the propagation velocity of the S-waves v_s and the density ρ: $l_0 = H$, $v_0 = v_s$ and $\rho_0 = \rho$ as the fundamental units; Poisson's ratio $\nu = 0.33$.

For open waveguides we also monitored that the energy balance condition[7] is satisfied and we compared this with the results obtained by other researchers and by other methods. Thus, in Fig. 2 we show the spatial dependence of the amplitude of the vertical displacements $|u_z(x, 0)|$ on the surface of an elastic layer of unit thickness ($H = 1$) with a circular cylindrical cavity, the surface of which

$$S_\tau: (x - x_c)^2 + (z - z_c)^2 = a^2$$

$$(x_c = 2,\ z_c = -0.4,\ a = 0.15)$$

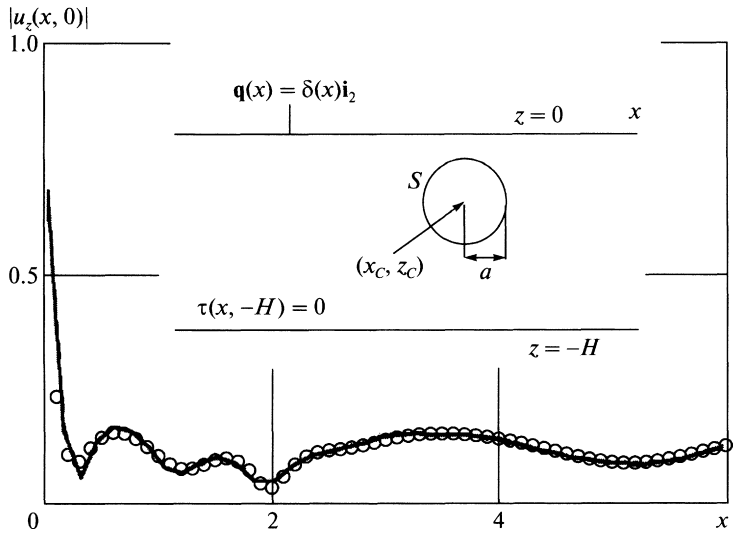

Fig. 2.

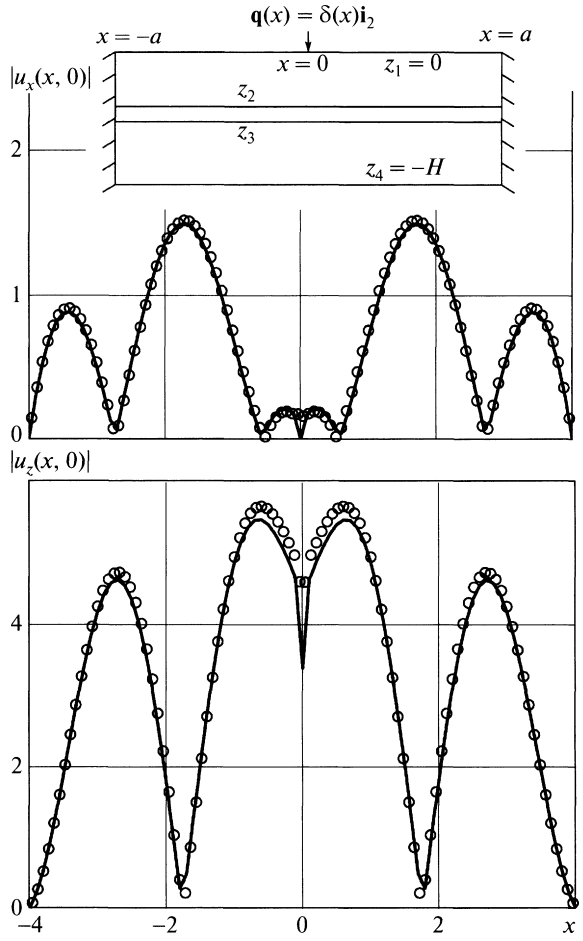

Fig. 3.

is stress-free (in the second condition (2.6) $\mathbf{p} \equiv 0$). The source of oscillations is a vertical concentrated surface load $\mathbf{q} = \delta(x)\mathbf{i}_2$; $\omega = \pi$. The continuous curve represents the results obtained by the rod element method,[11] and the markers represent the results obtained by the layered element method (the collocation method, $N = 32$).

A three-layer packet. We will consider a rectangular triple-layer elastic strip ($M = 3$), occupying the region

$$|x| \le a, \quad -H \le z \le 0, \quad z_2 = -0.4, \quad z_3 = -0.5$$

which models a plane sheet of laminated glass with a thin adhesive interlayer, deformed due to the action of a concentrated vertical load $\tau(x, 0) = \delta(x)\mathbf{i}_2$; the sides of the laminate are rigidly clamped: $\mathbf{u}(\pm a, z) = 0$. A characteristic feature of this example is the sharp difference between the elastic properties of the glass (layers D_1 and D_3 with $\nu_p = 1.68$, $\nu_s = 1$ and $\rho = 1$) from the properties of the adhesive (D_2 with $\nu_p = 0.0091$, $\nu_s = 0.0038$ and $\rho = 0.48$), and also the relative dimensions of the region $a/H \gg 1$ and the thinness of the interlayer ($h_2/H \ll 1$).

In Fig. 3 we show the amplitudes of the horizontal displacements of the surface of the laminate $|u_x(x, 0)|$ and the vertical displacements $|u_z(x, 0)|$ obtained using the layered element method ($N = 36$ elements, the continuous curves) and the finite element method (the FEMLAB software package: www.comsol.com, $N = 40{,}000$, the markers); $\omega = 0.5$ and $a = 4$.

It should be noted that, although the results obtained using the finite element method for the stress-strain state of the glass become stabilized when the number of elements $N \ge 16{,}000$, in the adhesive interlayer they are not established even when $N = 40{,}000$, and when the relative length a/H or the frequency ω increase numerical stability is also lost in layers D_1 and D_3. At the same time, due to the fact that each layered element equally accurately describes the stress-strain state at all internal points, the accuracy of the solution obtained using the layered element method in the interlayer D_2, is the same in the outer layers for any ratio of their elastic properties and dimensions.

Spatial scattering by a rigid inclusion. As an example of the realization of the layered element method in a three-dimensional formulation, we show in Fig. 4 the distribution of the amplitude of the vertical displacements $|u_z(x, y, 0)|$ of the surface of an elastic layer of thickness $H = 1$, containing a fixed disc-shaped rigid inclusion with surface

$$S_u : x^2/a^2 + y^2/b^2 + (z - z_c)^2/c^2 = 1 \quad (a = b = 0.7, \quad c = 0.1, \quad z_c = -0.5)$$

caused by a vertical concentrated load applied at the point $x_0 = y_0 = -6$, at a frequency $\omega = 2$. Here we have used $N = 180$ collocation points and three-dimensional layered elements, a distance $\delta = 0.02$ from S_u. The displacements of the surface above the inclusion are a minimum, and a dark zone is observed behind the inclusion.

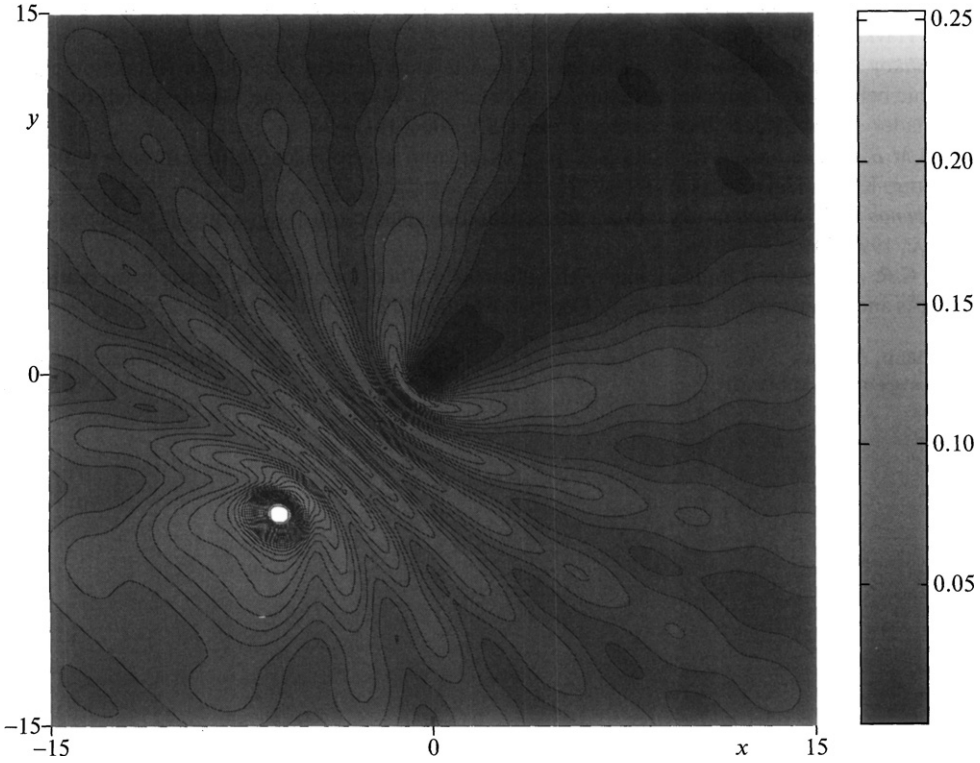

Fig. 4.

5. Conclusion

The proposed layered element method gives a physically clear wave solution of dynamic problems for bounded laminated samples and waveguides with obstacles. Although its realization also requires more considerable preliminary analytical work than when using the finite element, the boundary element or the finite difference methods, this is compensated by a considerable reduction in the number of elements required to achieve the same accuracy and by a corresponding reduction in the level of computing costs. Moreover, the use of exact analytical representations for the stress-strain state at internal points of the region and for travelling waves, departing to infinity, makes the use of the layered element method particularly suitable for open waveguides and for laminated structures with a sharp gradient of the stress-strain state at internal interfaces with considerably differing elastic properties.

Acknowledgements

This research was supported by the International Association for Promoting Cooperation with Scientists from the New Independent States of the Former Soviet Union (INTAS-05-1000008-7979) and the Minobrnauk Analytical Special Purpose Programme Department 2.1.1/1231.

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Translated by R.C.G.