Journal of Applied Mathematics and Mechanics 75 (2011) 56–64

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The excitation of waves by piezoelectric patch actuators arranged symmetrically on both surfaces of an elastic layer☆

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ARTICLE INFO

Article history:
Received 4 December 2009

ABSTRACT

An electromechanical system consisting of an elastic waveguide and flexible symmetrically arranged piezoelectric patch actuators attached to both of its surfaces is considered. In the mathematical model employed, which takes into account both the dynamic contact interaction of the patch with the waveguide and the presence of higher modes of oscillation of the layer, the effect of the geometrical and physical parameters of the system on the amount of energy delivered by the piezoelectric actuators in the substrate and its distribution between the excited Lamb waves is investigated. The analysis is carried out using the solution of a system of integro-differential equations, to which the boundary-value problem considered is reduced. In particular, it is shown that the maximum radiation of energy, transferred by antisymmetric and symmetric normal modes, is reached when the width of the patch is equal to a half-integer number of wavelengths of one of the normal modes of the patch-layer-patch triple-layer structure.

Electromechanical systems with piezoelectric actuators and sensors, constructed in the form of thin and flexible surface patches or internal layers in waveguide structures, are the basis of modern automatic control systems, which dynamically change the mechanical properties of materials and structures.

Since piezoelectric patch actuators are often employed to observe the state of thin-walled structural components, models have mainly been developed for low-frequency flexirural and longitudinal oscillations, excited by them in beams, plates and shells (see, for example, the review in Ref. 1). However, these models do not work when the wavelength is comparable with the waveguide thickness and do not enable one to describe the excitation of higher modes. This leads to the need to set up more complex models, in which higher modes of the elastic waveguide and the dynamic contact interaction of the flexible deformable patches attached to it are taken into account. The contact interaction of surface piezoelectric patch actuators with an elastic half-space has been analysed (see, for example, Refs 2-6). The case of a waveguide of finite thickness was considered.[7–9] A detailed analysis of the results obtained up to the present time can be found in Ref. 10.

The excitation of travelling waves by thin flexible piezoelectric patch actuators attached to one of the surfaces of an elastic waveguide or to the surface of a half-space was investigated in Refs 8 and 9 using an integral approach,[11] and the problem of generating selective directional radiation of the required mode in it was considered. In particular, resonance modes of energy radiation were found, which arise when the width of the patch actuator is equal to a half-integer number of wavelengths of one of the waves, excited in a double-layer patch-elastic waveguide structure.

Continuing these investigations, the previously developed approach is here extended to the case of symmetrically arranged piezoelectric elements on both surfaces of an elastic layer (Fig. 1), which is often encountered in practical applications. This arrangement enables both symmetrical and antisymmetrical oscillations to be excited by applying an electrical load to the patch actuators in-phase or antiphase respectively. Below, as far as possible, we will retain the notation previously employed and we will only omit some details of the general method of solution. For brevity, we will consider the example of a single pair of patch actuators; the solution obtained can be extended to the case of several pairs of patch actuators as was done previously in Refs 8 and 9.

In engineering practice, the generally accepted model[12] is considered, in which the amplitude-frequency characteristics of the Lamb waves excited by the piezoelectric actuator are determined without solving the contact problem. In this model the action of the patch actuator is modelled by a pair of concentrated shear forces, applied to the surface of the waveguide on the edges of the actuator. This

☆ *Prikl. Mat. Mekh.* Vol. 75, No. 1, pp. 83–94, 2011.

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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2011.04.008

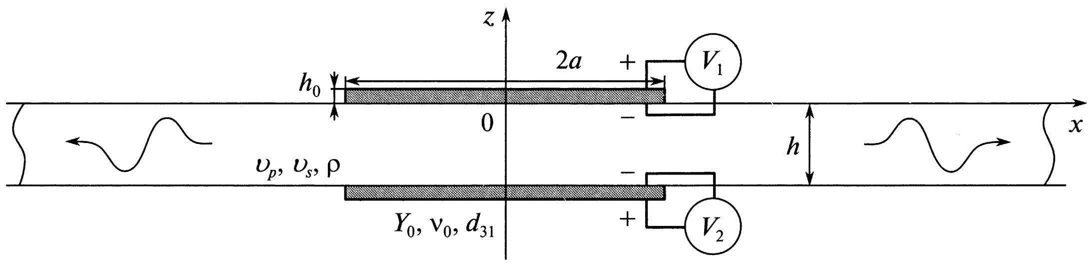

Fig. 1.

model is not suitable for analysing the stress-strain state in the contact area, but gives the correct relation between the amplitudes (and, correspondingly, the energies) of the excited fundamental modes in the frequency band up to the occurrence of the third (the first higher) mode. This is indicated, in particular, by a comparison with the numerical results obtained by solving the contact problem for a thin flexible actuator.[9] Obviously models in which the action of the patch on the substrate is described in terms of the solution of the contact problem considerably extend the frequency range of their applicability, thereby making it possible to carry out a wave-energy analysis in the case when several higher modes are excited.

In this paper we use the same model of the piezoelectric actuator as previously in Refs 8 and 9. The equations of the longitudinal (i.e., tangential to the waveguide surface) displacements and deformations of the patches are derived from the general equations of coupled linear electroelasticity, using certain conditions and constraints. The main one of these is the assumption that the thickness of the patch actuator is relatively small compared with its width, which enables us to regard it as a flexible strip in which the transverse stresses are negligibly small compared with the longitudinal stresses. These assumptions enable us to simplify the model while preserving its main advantage compared with the model described in Ref. 12, namely, the contact interaction of the elastic-deformed patches with the deformable substrate is taken into account. The simplifying assumptions and the limitations of the range of applicability following from these are described and discussed in more detail below.

1. The mathematical model of the interaction of a piezoelectric actuator with an elastic waveguide

We will consider, in the two-dimensional formulation (plane deformation), a uniform isotropic elastic layer of thickness h: $-\infty < x < \infty$, $-h \le z \le 0$, to the upper and lower surfaces of which flexible piezoelectric patch actuators of thickness h_0 and width $2a$ are attached (Fig. 1). The material of the patch actuators belongs to the 8 *mm* symmetry class,[13] where, for the upper actuator it is polarized in the direction of the z axis, while for the lower one it is polarized in the opposite direction. Outside the contact region $|x| \le a$, $z = 0$ and $|x| \le a$, $z = -h$ the surfaces of the waveguide are stress-free. Steady harmonic vibrations in the layer $\mathbf{u}(x, z)e^{-i\omega t}$ ($\mathbf{u} = (u = u_x, u_z)^T$ is the complex amplitude of the displacement vector) are excited by the unknown contact voltages $\mathbf{q}_1 e^{-i\omega t}$ and $-\mathbf{q}_2 e^{-i\omega t}$, which occur under the upper and lower actuators as a result of their deformation when an alternating electric potential difference $\varphi_n = \pm(V_n/2)e^{-i\omega t}(n = 1, 2)$ is applied to the electrodes. The harmonic factor $e^{-i\omega t}$ will henceforth be omitted and the system will be described in terms of the complex amplitudes of the corresponding quantities.

The wave field $u(x, z)$, excited in the elastic layer by the pair of actuators considered, satisfies Lamé's equations

$$(\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \mu\Delta\mathbf{u} + \rho\omega^2\mathbf{u} = 0, \quad |x| < \infty, \quad -h \le z \le 0 \tag{1.1}$$

and the boundary conditions

$$\boldsymbol{\sigma}|_{z=0} = \mathbf{q}_1, \quad \boldsymbol{\sigma}|_{z=-h} = -\mathbf{q}_2, \quad -\infty < x < \infty \tag{1.2}$$

Here λ and μ are the Lamé constants of the layer, ρ is its density and $\boldsymbol{\sigma} = (\tau_{xz}, \sigma_z)^T$ is the stress vector on the horizontal surface. We must also satisfy the radiation conditions, which follow from the limit-absorption principle.[11]

Boundary-value problem (1.1), (1.2) is open, since unknown contact voltages $\mathbf{q}_n = (q_{x,n}, q_{z,n})^T (n = 1, 2)$ occur in the boundary conditions. They must be determined from the simultaneous solution of the equations of motion for the layer and the patch actuators, connected by common contact conditions.

The displacement vectors $\mathbf{u}_n = (u_{x,n}, u_{z,n})^T$ for the upper actuator ($n = 1$) and the lower actuator ($n = 2$) satisfy the equations, which follow from the equations of motion:

$$\frac{\partial\sigma_{x,n}}{\partial x} + \frac{\partial\tau_{xz,n}}{\partial z} + \rho_0\omega^2 u_{x,n} = 0, \quad \frac{\partial\tau_{xz,n}}{\partial x} + \frac{\partial\sigma_{z,n}}{\partial z} + \rho_0\omega^2 u_{z,n} = 0 \tag{1.3}$$

the equations of electrostatics

$$\frac{\partial D_{x,n}}{\partial x} + \frac{\partial D_{z,n}}{\partial z} = 0 \tag{1.4}$$

and the equations of state, which express the relation between the mechanical stresses and strains and the electric field strength, and also the electric induction:

$$\begin{aligned} &\sigma_{x,n} = c_{11}\varepsilon_{x,n} + c_{13}\varepsilon_{z,n} - e_{31}E_{z,n}, \quad \sigma_{z,n} = c_{13}\varepsilon_{x,n} + c_{33}\varepsilon_{z,n} - e_{33}E_{z,n} \\ &\tau_{xz,n} = c_{44}\gamma_{xz,n} - e_{51}E_{x,n} \end{aligned} \tag{1.5}$$

$$D_{x,n} = e_{51}\gamma_{xz,n} + \epsilon_{11}E_{x,n}, \quad D_{z,n} = e_{13}\varepsilon_{x,n} + e_{33}\varepsilon_{z,n} + \epsilon_{33}E_{z,n} \tag{1.6}$$

Here $\sigma_{x,n}$, $\tau_{xz,n}$, $\sigma_{z,n}$ and $\varepsilon_{x,n}$, $\gamma_{xz,n}$, $\varepsilon_{z,n}$ are the components of the stress and strain tensors in the actuators while $D_{x,n}$, $D_{z,n}$ and $E_{x,n}$, $E_{z,n}$ are the components of the electric field strength and the electric induction vectors in them, c_{ij}, e_{ij} and ϵ_{ij} are the elastic moduli, the piezoelectric constants and the permittivity coefficients of the material of the actuator and ρ_0 is its density.

Equations (1.3)–(1.6) are reduced to equations in terms of the components of the displacement vectors u_n and the electric potentials φ_n when the Cauchy relations

$$\varepsilon_{x,n} = \frac{\partial u_{x,n}}{\partial x}, \quad \gamma_{xz,n} = \frac{\partial u_{x,n}}{\partial z} + \frac{\partial u_{z,n}}{\partial x}, \quad \varepsilon_{z,n} = \frac{\partial u_{z,n}}{\partial z} \tag{1.7}$$

and the expressions for the electric field strength

$$E_{x,n} = -\frac{\partial\varphi}{\partial x}, \quad E_{z,n} = -\frac{\partial\varphi}{\partial z} \tag{1.8}$$

are substituted into them.

The boundary conditions (written here, for brevity, solely for the upper actuator) consist of the conditions for there to be no stresses on its outer surface

$$\tau_{xz,1}|_{z=h_0} = 0, \quad \sigma_{z,1}|_{z=h_0} = 0, \quad -a \le x \le a \tag{1.9}$$

$$\sigma_{x,1}|_{x=\pm a} = 0, \quad \tau_{xz,1}|_{x=\pm a} = 0, \quad 0 \le z \le h_0 \tag{1.10}$$

and the continuity of the displacement and stress field $\boldsymbol{\sigma}_1 = (\tau_{xz,1}, \sigma_{z,1})^T$ in the region where the actuator is in contact with the layer

$$\boldsymbol{\sigma}_1|_{z=0} = \mathbf{q}_1, \quad \mathbf{u}_1|_{z=0} = \mathbf{u}|_{z=0}, \quad -a \le x \le a \tag{1.11}$$

To these we must add the conditions on the horizontal electrode surfaces of the actuator

$$\varphi_1|_{z=h_0} = +V_1/2, \quad \varphi_1|_{z=0} = -V_1/2, \quad -a \le x \le a \tag{1.12}$$

and on the faces of the piezoelectric elements

$$D_{x,1}|_{x=\pm a} = 0, \quad 0 \le z \le h_0 \tag{1.13}$$

The thickness of the piezoelectric actuators used in practice, as a rule, is considerably less than their horizontal dimension ($h_0 \ll 2a$), i.e., the longitudinal extension compression deformations ε_x and the related stresses σ_x predominate. Moreover, we will not consider the high-frequency band, in which the characteristic wavelength λ_0 would be commensurable with the thickness h_0. Hence, to simplify the equations of motion of the actuators we will use the following assumptions:

1) the components of the displacements and stresses $u_{x,n}$, $u_{z,n}$ and $\sigma_{x,n}$ are constant over the thickness of the patch actuator (which is applicable in the frequency band in which $h_0 < \lambda_0/8$);
2) the electric field is constant over the width of the patch and is directed perpendicular to its surface: $E_{x,n} = 0$, $E_{z,n} = V_n/h_0$;
3) the patch is relatively thin $h_0 \ll a$, and hence does not resist bending, i.e., in the contact area $\sigma_z = 0$.

The use of the first approximation enables us to integrate the first of the equations of motion (1.3) over the thickness of the actuator and reduces it to the form

$$\frac{\partial\sigma_{x,n}}{\partial x} + \rho_0\omega^2 u_{x,n} = \frac{1}{h_0}q_{x,n} \tag{1.14}$$

It enables us to simplify the equation of state (1.5):

$$\sigma_{x,n} = c_{11}\varepsilon_{x,n} + e_{31}E_{z,n} \tag{1.15}$$

Hence, taking the first of conditions (1.10) and the second assumption into account, we obtain

$$\frac{d^2 \upsilon_n}{dx^2} + \kappa_0^2\upsilon_n = b_0 q_n, \quad |x| \le a; \quad \left.\frac{d\upsilon_n}{dx}\right|_{x=\pm a} = e_n; \quad n = 1, 2 \tag{1.16}$$

where

$$\kappa_0^2 = \rho_0\omega^2/c_{11}, \quad b_0 = 1/(c_{11}h_0), \quad e_n = e_{31}E_{z,n}/c_{11}$$

$\upsilon_n = u_{x,n}$ are the longitudinal displacements in the patch actuators and $q_n = q_{x,n}$ are the unknown contact shear stresses. The third assumption enables us to simplify the patch-layer contact conditions (1.11) and reduce them to the form

$$\tau_{xz,1}|_{z=0} = q_1, \quad \tau_{xz,2}|_{z=-h} = -q_2, \quad u_{x,1}|_{z=0} = \upsilon_1, \quad u_{x,2}|_{z=-h} = \upsilon_2, \quad -a \le x \le a \tag{1.17}$$

2. The system of integral equations

For the displacements **u**, produced in the elastic layer by the surface loads, defined by the first two conditions of (1.17), the following integral representation in the form of a convolution of Green's matrix with the surface stresses q_n or, in equivalent form, an inverse Fourier transformation of the product of their Fourier-symbols, holds[11]

$$\mathbf{u}(x,z)=\sum_{n=1}^{2}\int_{-a}^{a}\mathbf{k}_n(x-\xi,z)q_n(\xi)d\xi=\frac{1}{2\pi}\sum_{n=1}^{2}\int_{\Gamma}\mathbf{K}_n(\alpha,z)Q_n(\alpha)e^{-i\alpha x}d\alpha \tag{2.1}$$

Here

$$\mathbf{K}_n=(K_{n,11},K_{n,21})^T=\mathcal{F}_x[\mathbf{k}_n],\quad Q_n=\mathcal{F}_x[q_n]$$

$\mathbf{k}_n=(k_{n,11},k_{n,21})^T (n=1,2)$ are the first columns of Green's matrix of the elastic layer with the load, applied to the upper and lower boundaries respectively, and is the Fourier transformation operator with respect to the x variable. The contour of integration Γ passes along the real axis, deviating from it into the complex plane when it passes around the real poles of the integrands in accordance with the limit-absorption principle.

The elements of the Fourier-symbols of Green's vector functions in the case considered can be written in the explicit form

$$\mathbf{K}_1(\alpha,z)=\frac{1}{\Delta(\alpha)}\begin{Vmatrix}-iM_1(\alpha,z)\\ \alpha S_1(\alpha,z)\end{Vmatrix},\quad \mathbf{K}_2(\alpha,z)=\frac{1}{\Delta(\alpha)}\begin{Vmatrix}-iM_1(\alpha,-z-h)\\ -\alpha S_1(\alpha,-z-h)\end{Vmatrix}$$

(for the form of the functions M_1, S_1 and Δ see, for example, Ref. 8).

Substituting representation (2.1) into the displacement continuity conditions, defined by the third and fourth conditions of (1.17), we obtain the following system of integral equations

$$\mathcal{K}\mathbf{q}=\mathbf{v},\quad |x|\le a \tag{2.2}$$

Here

$$\mathbf{q}=\begin{Vmatrix}q_1\\ q_2\end{Vmatrix},\quad \mathbf{v}=\begin{Vmatrix}v_1\\ v_2\end{Vmatrix},\quad \mathcal{K}=\begin{Vmatrix}\mathcal{K}_1 & \mathcal{K}_2\\ \mathcal{K}_2 & \mathcal{K}_1\end{Vmatrix}$$

$$\mathcal{K}_j q_n\equiv\int_{-a}^{a}k_j(x-\xi)q_n(\xi)d\xi=\frac{1}{2\pi}\int_{\Gamma}K_j(\alpha)Q_n(\alpha)e^{-i\alpha x}d\alpha,\quad j=1,2$$

$$K_1(\alpha)=-iM_1(\alpha,0)/\Delta(\alpha),\quad K_2(\alpha)=-iM_1(\alpha,-h)/\Delta(\alpha)$$

In view of the fact that, as $\alpha\to\infty$, the function $K_1(\alpha)$ decreases as α^{-1}, while $K_2(\alpha)$ decreases exponentially, the diagonal operator K_1 is singular with a logarithmic singularity of the kernel $k_1(x)$ (as in contact problems for a rigid punch on an elastic base), while K_2 is an operator with a smooth kernel, which describes the mutual effect of the actuators.

Since the right-hand side **v** of the system of equations obtained is an unknown vector function, the problem can only be solved by simultaneous solution of Eqs (1.16) and (2.2), taking the boundary conditions in problem (1.16) into account.

3. Discretisation of the problem

To solve the integro-differential problem obtained numerically we will use the method proposed earlier in Ref. 14 of reducing Wiener-Hopf integral equations with a meromorphic symbol of the kernel to an infinite system of linear algebraic equations. Following the general scheme of the method, the matrix integral operator (2.2) is underdefined over the whole real axis of the unknown vector-function

$$\boldsymbol{\varphi}(x)=\mathcal{K}\mathbf{q},\quad |x|>a \tag{3.1}$$

which enables us to apply a Fourier transformation with respect to x to it. As a result we arrive at the following matrix functional equation

$$\mathbf{KQ}=\mathbf{V}+\boldsymbol{\Phi};\quad \mathbf{Q}=\mathcal{F}_x[\mathbf{q}],\quad \mathbf{V}=\mathcal{F}_x[\mathbf{v}],\quad \boldsymbol{\Phi}=\mathcal{F}_x[\boldsymbol{\varphi}] \tag{3.2}$$

in terms of the Fourier-symbols of the corresponding functions.

Closure of the contour Γ in the integrals which occur in the definition of the operator K (see formula (2.1)), and replacement of their sums by residues, leads to the representation of the function $\boldsymbol{\varphi}(x)$ in the form of series

$$\boldsymbol{\varphi}(x)=\sum_{k=1}^{\infty}\mathbf{s}_k^{\pm}e^{\pm i\zeta_k(x\mp a)},\quad |x|>a \tag{3.3}$$

with unknown constant coefficients

$$\mathbf{s}_k^{\pm}=\left(t_{1k}^{\pm}+\frac{r_{2k}}{r_{1k}}t_{2k}^{\pm},\frac{r_{2k}}{r_{1k}}t_{1k}^{\pm}+t_{2k}^{\pm}\right)^T \tag{3.4}$$

in which

$$t_{nk}^{\pm} = ir_{1k}Q_n(\mp\zeta_k)e^{i\zeta_k a}, \quad r_{nk} = \operatorname{res} K_n(\alpha)|_{\alpha=\zeta_k}; \quad n = 1, 2$$

and ζ_k is the pole of the function $K_n(\alpha)$ (the zeros of the denominator $\Delta(\alpha)$), situated above the contour Γ; the upper signs are taken for $x > a$ and the lower signs for $x < -a$. Consequently,

$$\mathbf{\Phi} = \mathcal{F}_x[\varphi] = \mathbf{\Phi}^+e^{i\alpha a} + \mathbf{\Phi}^-e^{-i\alpha a}, \quad \mathbf{\Phi}^{\pm} = \pm i\sum_{k=1}^{\infty}\frac{\mathbf{s}_k^{\pm}}{\alpha \pm \zeta_k} \tag{3.5}$$

Equation (3.2) contains three unknown vector functions: Q, V and Φ. To eliminate one of these we use the relation which is obtained after applying a Fourier transformation to Eqs. (1.16), underdefined along the whole x axis by zero. Taking into account the rule for a Fourier transformation of the derivatives of discontinuous functions, we have

$$\mathbf{V} = b_0G\mathbf{E}\mathbf{Q} + \mathbf{\Phi}_0 - \mathbf{F} \tag{3.6}$$

Here $G = G(\alpha) = 1/(-\alpha^2 + \kappa_0^2)$ is the Fourier symbol of the fundamental solution of Eq. (1.16) and **E** is the unit 2×2 matrix,

$$\mathbf{\Phi}_0 = \mathbf{\Phi}_0^+e^{i\alpha a} + \mathbf{\Phi}_0^-e^{-i\alpha a}, \quad \mathbf{\Phi}_0^{\pm} = \pm i\alpha G\mathbf{s}_0^{\pm} \tag{3.7}$$

$$\mathbf{F} = \mathbf{F}^+e^{i\alpha a} + \mathbf{F}^-e^{-i\alpha a}, \quad \mathbf{F}^{\pm} = \pm G\mathbf{e} \tag{3.8}$$

$\mathbf{s}_0^{\pm} = (t_{10}^{\pm}, t_{20}^{\pm})^T$ is a column vector, composed of the unknown constants $t_{n0}^{\pm} = \upsilon_n(\pm a)(n = 1, 2)$ and $\mathbf{e} = (e_1, e_2)^T$ is the known column vector of the control parameter of the electromechanical system (see Eq (1.16)).

It follows from (3.2) and (3.6) that

$$\mathbf{Q} = (\mathbf{K} - b_0G\mathbf{E})^{-1}(\mathbf{\Phi}_0 + \mathbf{\Phi} - \mathbf{F}) \tag{3.9}$$

Here the inverse matrix is written in explicit form

$$(\mathbf{K} - b_0G\mathbf{E})^{-1} = \frac{\sigma_0^2}{\Theta}\mathbf{L}$$

$$\mathbf{L} = \begin{Vmatrix} -i\sigma_0^2M_1(\alpha, 0) + b_0\Delta & i\sigma_0^2M_1(\alpha, -h) \\ i\sigma_0^2M_1(\alpha, -h) & -i\sigma_0^2M_1(\alpha, 0) + b_0\Delta \end{Vmatrix}$$

$$\Theta = \sigma_0^4\Theta_0 - 2ib_0\sigma_0^2M_1(\alpha, 0) + b_0^2\Delta, \quad \Theta_0 = -\frac{\sigma_2^2\kappa_2^4}{8\mu}\operatorname{sh}\sigma_1h\operatorname{sh}\sigma_2h$$

$$\sigma_n^2 = \alpha^2 - \kappa_n^2, \quad n = 0, 1, 2 \tag{3.10}$$

where κ_0 is the wave number of the longitudinal vibrations of the piezoelectric patch actuator (see Eqs (1.16)), and κ_1 and κ_2 are the wave numbers of the longitudinal and transverse waves of the elastic layer.

The vector function Q is wholly in the complex plane α (like the Fourier transformation of the vector function **q**, specified in the limited region $|x| \le a$), and hence the poles $\pm z_l$ and $\pm\zeta_k$, guided by the zeros of the functions $\Theta(\alpha)$ and $\Delta(\alpha)$, introduced into the numerator of representation (3.9), must be eliminated. To do this it is necessary to satisfy the conditions

$$\begin{aligned} &\mathbf{L}(\pm z_l)\big(\mathbf{\Phi}_0(\pm z_l) + \mathbf{\Phi}(\pm z_l) - \mathbf{F}(\pm z_l)\big) = 0, \quad l = 1, 2, \ldots \\ &\mathbf{L}(\pm\zeta_k)\operatorname{res}\mathbf{\Phi}|_{\alpha=\pm\zeta_k} = 0, \quad k = 1, 2, \ldots. \end{aligned} \tag{3.11}$$

In view of the equality det**L** = Q, the rows of the matrices $\mathbf{L}(\pm z_l)$ and $\mathbf{L}(\pm\zeta_k)$ are linearly dependent, and hence, for fixed values of l and k, each pair of conditions (3.11) gives only one independent relation. Their combination is an infinite system of linear algebraic equations in the column vectors $\mathbf{s}_k = (\mathbf{s}_k^+, \mathbf{s}_k^-)$. Further steps (the derivation of the final form of the elements of the matrix of the system and its regularization by taking into the account the asymptotics of the unknowns s_k as $k \to \infty$) do not differ in principle from those described earlier in Refs 8 and 14.

After finding the unknown $\mathbf{s}_k^{\pm}$, the Fourier-symbol of the column vector of the unknown contact stresses Q, taking relations (3.5)-(3.9) into account, can be rewritten in the form

$$\mathbf{Q} = (\mathbf{K} - b_0G\mathbf{E})^{-1}\left(\mathbf{C}^+e^{i\alpha a} + \mathbf{C}^-e^{-i\alpha a}\right); \quad \mathbf{C}^{\pm} = \mathbf{\Phi}_0^{\pm} + \mathbf{\Phi}^{\pm} - \mathbf{F}^{\pm} \tag{3.12}$$

The components of the vector function $C^{\pm}$ have a non-exponential behaviour in the complex plane, which enables Jordan's lemma and Cauchy's theorem to be used to represent the unknown contact stresses

$$\mathbf{q} = \mathcal{F}_x^{-1}[\mathbf{Q}] \equiv \frac{1}{2\pi}\int_{\Gamma}\mathbf{Q}(\alpha)e^{-i\alpha x}d\alpha$$

in the form of the series

$$\mathbf{q}(x)=\sum_{l=1}^{\infty}(\mathbf{p}_l^{+}e^{iz_l(x+a)}+\mathbf{p}_l^{-}e^{-iz_l(x-a)}) \tag{3.13}$$

where

$$\mathbf{p}_l^{\pm}=(p_{1l}^{\pm},p_{2l}^{\pm})^T=\operatorname{res}(\mathbf{K}-b_0G\mathbf{E})^{-1}|_{\alpha=z_l}\,\mathbf{C}^{\mp}(\mp z_l)$$

To do this we first split the vector-function Q into two integrals, corresponding to its exponential behaviour (3.12) and the closure of the contour Γ on the side on which the exponent decreases. The poles $\pm z_l$ in this case becomes irremovable and make a contribution to the form of the sum of the residues (3.13).

Remark. The scheme described can also be used for elastic multilayer waveguides, for example, when modelling the interaction of the patch actuator with laminated composite materials. To do this one only requires the symbols of Green's matrices $K(\alpha, z)$ and the poles z_l and ζ_k, the algorithms for constructing and searching for which are well developed at the present time.

4. Numerical analysis: the structure of the wave fields and resonances

After finding the contact stresses $\mathbf{q}$, the wave field $\mathbf{u}$ in the layer is determined from integral representation (2.1). For $|x|>a$ the closure of the Γ contour in the integrals present here and their replacement by the sums of residues leads to the representation of u in the form of an expansion in normal modes of the elastic layer:

$$\mathbf{u}(x,z)=\sum_{k=1}^{\infty}\sum_{n=1}^{2}t_{nk}^{\pm}\mathbf{a}_{nk}^{\pm}(z)e^{\pm i\zeta_k(x\mp a)},\quad |x|>a;\quad \mathbf{a}_{nk}^{\pm}(z)=\left.\frac{\mathbf{K}_1(\alpha,z)}{K_1(\alpha)}\right|_{\alpha=\mp\zeta_k} \tag{4.1}$$

The upper signs in each of the terms are taken for $x>a$ and the lower signs for $x<-a$. Then travelling waves, departing to infinity without attenuation with phase and group velocities $c_{p,k}=\omega/\zeta_k$ and $c_{g,k}=d\omega/d\zeta_k$ correspond to the first N_1 real poles ζ_k in the expansion, whereas exponentially attenuating waves correspond to the complex poles ζ_k.

When $|x|\le a$ it is first necessary to split each of the integrals in relations (2.1) into two corresponding to the exponential behaviour of the column vector (3.12), as was done when deriving formula (3.13), and then close the contour in the required direction and replace the integrals by the sums of residues. Finally, we arrive at a representation of the wave field in the layer between the actuators in the form of the superposition of travelling and attenuating waves with wave numbers z_l for the normal modes of a triple-layer actuator-layer-actuator structure

$$\mathbf{u}(x,z)=\sum_{l=1}^{\infty}(\mathbf{b}_l^{+}(z)e^{iz_l(x+a)}+\mathbf{b}_l^{-}(z)e^{-iz_l(x-a)}),\quad |x|\le a$$

$$\mathbf{b}_l^{\pm}(z)=\sum_{n=1}^{2}\mathbf{K}_n(\mp z_l,z)p_{nl}^{\pm} \tag{4.2}$$

Examples of the real branches of the dispersion relations of an elastic layer $\zeta_k(\omega)$ and a triple-layer actuator-layer-actuator structure $z_l(\omega)$ are shown in Fig. 2 for antisymmetrical modes (the upper part of Fig. 2) and symmetrical modes (the lower part). All the numerical results here and below are presented in dimensionless form, in units expressed in terms of three basic quantities: the thickness h (for the linear dimensions), the velocity of the S-waves v_s and the density ρ of the waveguide. Here the dimensionless angular frequency $\omega=2\pi fh/v_s$, where f is the dimensional frequency in hertz. The elastic properties of the layer when $v_s=1$ are determined by Poisson's ratio $\nu=0.3$. The dimensionless parameters of the actuators are

$$h_0=0.167,\quad Y_0=0.866,\quad \nu_0=0.3,\quad \rho_0=0.997$$

The parameter a is varied in the numerical examples.

In addition to the excited displacements $\mathbf{u}$, an important characteristic of an electromechanical system is the energy flux E_0 applied from the source (the actuator system) into the waveguide, averaged over the period of the oscillations. By replacing the contour integrals by the sum of residues we can represent the quantity E_0 in the form of the sum of averaged energy fluxes $E_k^{\pm}$, removed to the right and to the left to infinity by each of the travelling waves individually (for more detail see, for example, Refs. 8 and 11):

$$E_0=\sum_{k=1}^{N_1}E_k^{+}+\sum_{k=1}^{N_1}E_k^{-} \tag{4.3}$$

$$E_k^{\pm}=\frac{\omega}{4}\left(\operatorname{res}\mathbf{K}(\alpha)|_{\alpha=\zeta_k}\mathbf{Q}(\mp\zeta_k),\mathbf{Q}(\mp\zeta_k)\right),\quad k=1,2,\ldots,N_1 \tag{4.4}$$

It is important to note that, in view of the linearity of the problem, the contact stress vector can be represented in the form

$$\mathbf{q}=\frac{1}{2}(e_1-e_2)\mathbf{q}_A+\frac{1}{2}(e_1+e_2)\mathbf{q}_S;\quad \mathbf{q}_A=\begin{Vmatrix}q_A\\-q_A\end{Vmatrix},\quad \mathbf{q}_S=\begin{Vmatrix}q_S\\q_S\end{Vmatrix} \tag{4.5}$$

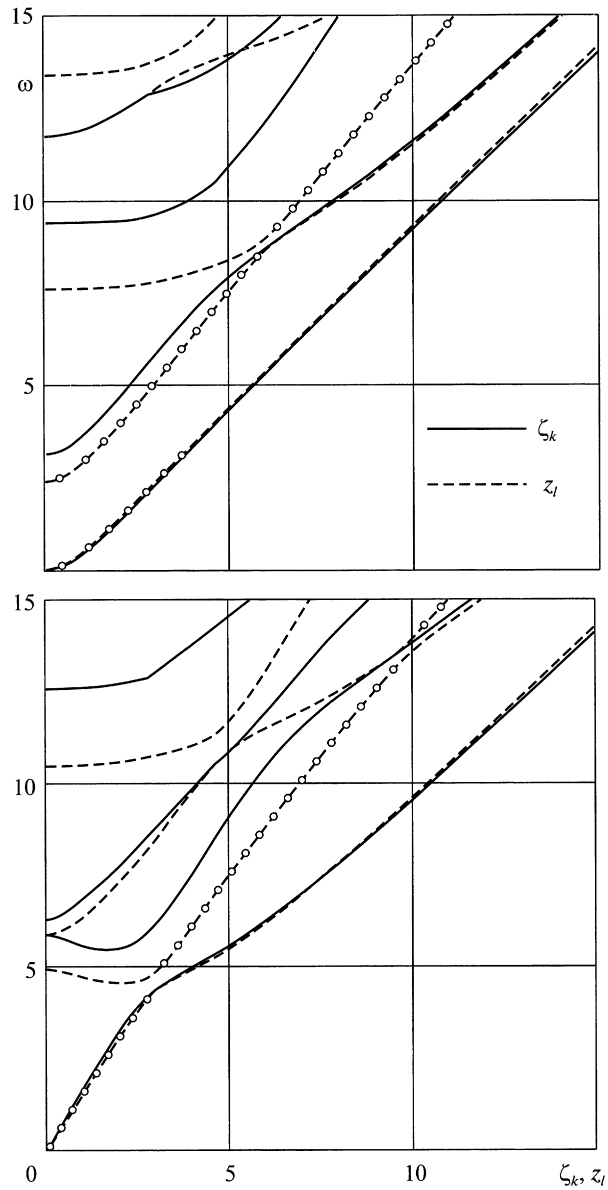

Fig. 2.

where $\mathbf{q}_A$ and $\mathbf{q}_S$ are common vectors of the contact stresses, which occur for antisymmetrical excitation ($e_1 = 1$, $e_2 = -1$) and symmetrical excitation ($e_1 = 1$, $e_2 = 1$) of the piezoelectric elements respectively. A similar splitting holds for the wave field u and hence

$$E_k^{\pm} = \frac{1}{4}|e_1 - e_2|^2 E_{k,A}^{\pm} + \frac{1}{4}|e_1 + e_2|^2 E_{k,S}^{\pm} \tag{4.6}$$

where $E_{k,A}^{\pm}$ and $E_{k,S}^{\pm}$ are the energy fluxes $E_k^{\pm}$ averaged over the vibration period, removed to the right and to the left to infinity by the k-th normal mode of the elastic layer for antisymmetrical and symmetrical excitation of the piezoelectric elements.

Hence, we have

$$E_0 = \frac{1}{4}|e_1 - e_2|^2 E_A + \frac{1}{4}|e_1 + e_2|^2 E_S; \quad E_A = \sum_{k \in A} E_{k,A}, \quad E_S = \sum_{k \in S} E_{k,S} \tag{4.7}$$

(summation is carried out over the numbers of antisymmetrical and symmetrical normal modes respectively). It follows from relations (4.6) and (4.7) that the behaviour of the system considered for the antisymmetrical and symmetrical methods of excitation of the actuators completely determines its properties for an arbitrary method of excitation.

One of the important problems in practice is the choice of the size of the actuator which will ensure maximum radiation of the wave energy E_0, E_A or E_S. Examples of the dependence of these quantities on the half-width of the actuator a for a fixed frequency $\omega = 2$ (the two-mode regime) and for $\omega = 5.5$ (four-mode radiation) are shown in Fig. 3. For $\omega = 2$ the relations E_A(a) and E_S(a) are strictly periodic, and

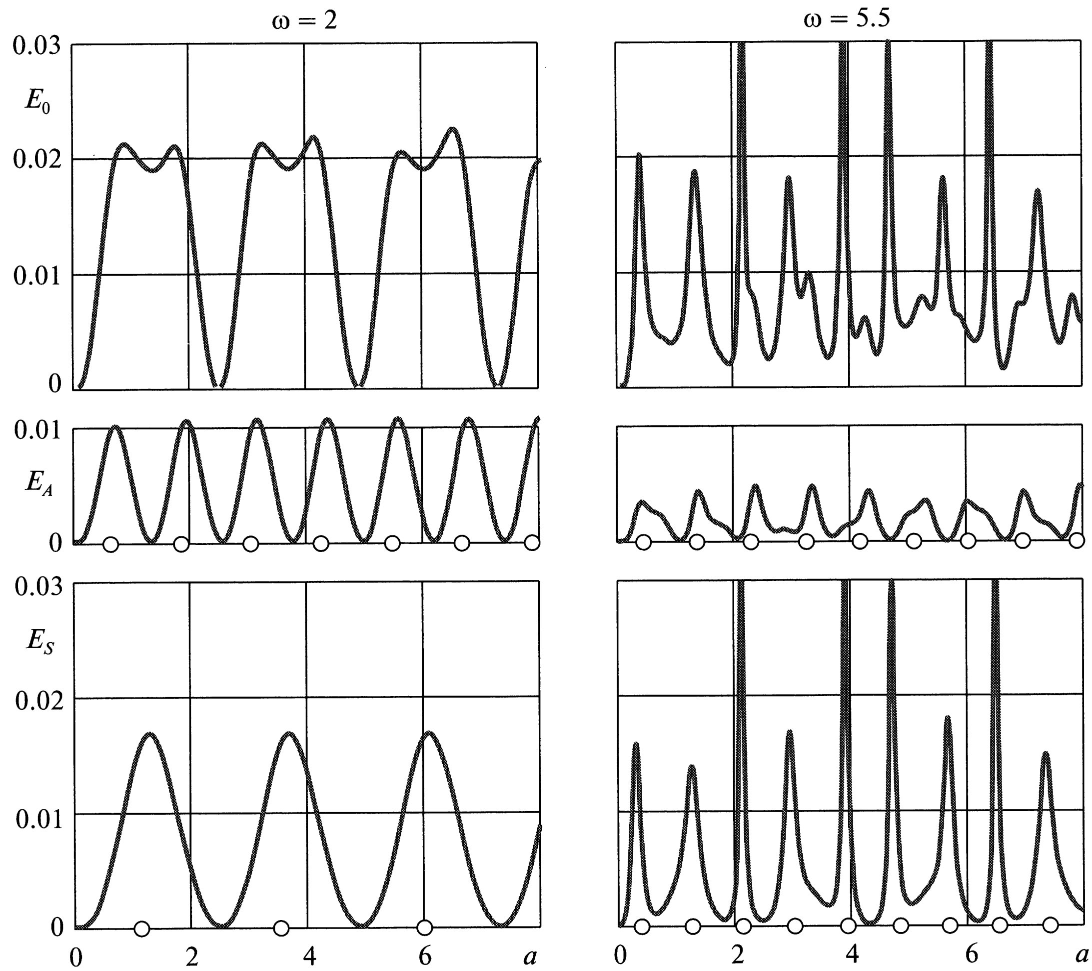

Fig. 3.

they reach their maximum values when

$$a = \lambda_l(k + 1/2)/2, \quad k = 0, 1, 2, \ldots \tag{4.8}$$

where $\lambda_l = 2\pi/\pi_l$ is the wavelength of the fundamental antisymmetrical mode a_0 or the symmetrical mode s_0 (for E_A or E_S respectively). The values of a, calculated from (4.8), are shown in Fig. 3 by the circle markers. As in the case considered previously, the positioning of the actuators on only one side of the waveguide,[8,9] and the occurrence of these maxima can be explained by the total addition of the amplitudes of the corresponding fundamental modes, excited by two point sources, situated on the edges of the actuators and operating in antiphase (i.e., in the approximation, employed in the model[12] where the contact stresses were represented by the expressions $q_n = e_n[\delta(x - a) - \delta(x + a)]$, $n = 1, 2$). This addition occurs when the distance between the sources $2a$ is equal to a half-integer number of wavelengths λ (condition (4.8). If the width of the actuator $2a$ is equal to an integer number of wavelengths, excited in antiphase, the total amplitude is equal to zero and, for the corresponding values of a, minima are observed on the graphs of $E_A(a)$ and $E_S(a)$.

The coincidence in the positions of the maxima and minima of the piezoelectric actuator power, obtained in the rigorous solution of the contact problem, for values of a in the two-point approximation of its action on the layer indicates the low sensitivity of this integral characteristic, like the radiated energy, to the specific form of the distribution of the contact stresses inside the contact area, if the inherent two-mode range predominant concentration of the contact stresses on the edges of the actuators is ensured. The applicability of the model[12] in the two-mode range, in particular, is explained by this.

The contact stresses have a root singularity at the edges of the contact area

$$q_n(x) \sim c/\sqrt{a^2 - x^2}, \quad x \to \pm a$$

for all ω. Hence, the mechanism of the addition or annihilation of the waves, excited by the edges of the actuators, continues to act at high frequencies also. However, when higher modes ($\omega > \pi$) appear, it becomes unclear which of them makes the main contribution to the flux of radiated wave energy, i.e., which of the poles z_l must be taken to determine the optimum dimension a from formula (4.8). A numerical analysis shows that, as for the actuators considered previously in Refs 8 and 9, as ω increases, the role of the fundamental mode transfers in turn from the fundamental mode to the next in order higher mode. The corresponding parts of the branches of the dispersion curves $z_l(\omega)$ are shown in Fig. 2 by the circle markers. As the frequency ω increases, they align themselves along a certain line, the direction of which, as previously,[8,9] is determined by the wave number of the Rayleigh-type surface wave in an elastic half-space, coated with the thin film considered. The values of a, calculated at the markers of the pole z_l indicated agree well with the maxima of the energy E_A and E_S (see Fig. 3).

It should be noted that exceptions to this general rule for determining the resonance dimensions of the actuators are encountered in narrow frequency bands in the neighbourhood of the cutoff frequencies, at which new real branches of the functions $z_l(\omega)$ appear. Usually, the number of real z_l and ζ_k, corresponding to antisymmetrical or symmetrical normal modes, is different (for example, in the case of antisymmetrical excitation, formula (4.8) does not give dimensions which agree with the resonance width of the actuator $2a$ in the range $2.3 < \omega < 3.2$).

Acknowledgement

This research was supported by the Analytical Department of the Special Purpose Programme (2.1.1/1231).

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Translated by R.C.G.